JOURNAL OF ALGEBRA 47, 244–267 (1977)

Stable Structure of Noncommutative Noetherian Rings

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Communicated by P. M. Cohn

Received January 5, 1976

The intention of this paper is to consider the following three theorems.

STABLE RANGE THEOREM. *Let R be a Noetherian ring with* Kdim $R = n$, *and suppose $R = \sum_1^{n+2} a_i R$. Then there exist $f_i \in R$ such that*

$$R = \sum_1^{n+1} (a_i + a_{n+2} f_i)R.$$

SERRE'S THEOREM. *Let R be as above and suppose that M is a "big" finitely generated R-module. Then $M \cong M' \oplus R$.*

CANCELLATION THEOREM. *Let R and M be as above and suppose $M \oplus R \cong N \oplus R$ for some module N. Then $M \cong N$.*

These results are of course well known and well studied in a commutative setting. In this paper, the Stable Range Theorem is proved for several classes of noncommutative rings and the other two theorems are proved for simple and related rings.

Section 1 gives some elementary results about semiprime rings that are used throughout the paper. In Section 2, several types of the Stable Range Theorem are proved, all of which follow from the same basic result (Proposition 2.1). The Stable Range Theorem itself is proved for rings known as right ideal invariant. Examples of such rings are given in Section 3 and include Noetherian rings which are either commutative, fully bounded, Asano, or semiprime and hereditary. Strangely, the proof of this result follows closely on the proof of [18, Lemma 1.4], which gives a bound on the number of generators of a right ideal in a simple ring. It is not surprising therefore that in the special case of simple rings, or indeed Asano orders, the Stable Range Theorem can be strangthened to give the same stability property for the generators of a right ideal (the Stable Range Theorem for Ideals).

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ISSN 0021-8693

The Stable Range Theorem for Ideals is crucial to the remainder of the paper and is used to prove both Serre's theorem and the Cancellation Theorem. In Section 4 these two theorems are proved for simple Noetherian rings and for certain modules of overrings of simple rings. For a simple ring R, Serre's theorem and the Cancellation Theorem are proved for finitely generated torsion-free modules of rank $\geqslant$ Kdim $R + 2$ (this number is loosely called the bound for the two theorems). Note that for simple rings we have not needed any of the projectivity conditions required in the commutative case. This is perhaps not surprising, since Serre's theorem and the Cancellation Theorem for torsion-free modules are closely connected with (and prove) the Stable Range Theorem for Ideals. Thus in Section 5, when these two theorems are proved for polynomial extensions of simple rings, we need to add a projectivity condition.

In Section 6 we restrict our attention to the Weyl algebra A_n. In this case we can obtain a dramatic improvement in the bound for all three theorems and indeed make it independent of the Krull dimension of A_n (which equals n for $A_n(k)$, k a field of characteristic zero). So in particular it is proved that projective modules of rank $\geqslant 5$ over A_n are free, a result that is probably best compared with Serre's question that asked whether all projective $k[x_1, \ldots, x_n]$ modules are free.

The final section proves Serre's theorem and the Cancellation Theorem for Asano orders. These parallel the corresponding results about simple rings although no information is given about overrings of Asano orders. In the special case of Dedekind prime rings this answers a question in [4].

Throughout this paper all rings are assumed to possess an identity and all modules to be unitary. Many of the results in this paper depend on Krull dimension and for the basic definitions and concepts involved the reader is referred to [5]. The right (left) Krull dimension of an R-module M is written r-$\mathrm{Kdim}_R M$ (respectively, l-$\mathrm{Kdim}_R M$) and both suffix and prefix are dropped when no ambiguity can arise. Conditions which are not preceded by either "left" or "right" are taken as two-sided conditions. So a Noetherian ring is a left- and right-Noetherian ring.

1. Semiprime Rings

This section establishes various technical results about semiprime rings that are required later. Essentially these results enable us to work with regular elements in the remainder of the paper and thus can be ignored if the reader wishes to work with domains.

Let R be semiprime right Goldie ring with full quotient ring Q and I a right ideal of R. Then I is said to be *uniform* if any two nonzero submodules of I have nonzero intersection. This is easily seen to be equivalent to IQ being a minimal right ideal of Q. Now Q is the direct sum of simple

Artinian rings and has finite length, say n, as a module over itself. Any direct sum of n minimal right ideals inside Q is equal to Q and so a right ideal of R is essential if and only if it contains a direct sum of n uniform ideals. (Recall that a right ideal is essential if and only if it contains a regular element.) In general, if the direct sum of k uniform right ideals is essential in I, then I is said to have *uniform dimension* k. Equivalently IQ is the direct sum of k minimal right ideals in Q.

LEMMA 1.1. *Let R be a prime right Goldie ring and $a, b \in R$. Then there exists $f \in R$ such that the uniform dimension of $(a + bf)R$ equals that of $aR + bR$.*

Proof. By induction it is sufficient to prove the lemma when bR is uniform. If $I = \text{r-ann}(a) = 0$, then a is regular and we are through. So assume $I \neq 0$. Then as R is prime, $bRI \neq 0$, so there exist $f \in R$ and $r \in I$ such that $bfr \neq 0$. We claim that $a + bf$ has the required uniform dimension. To show this, it is sufficient to prove that $(a + bf)R$ contains essential submodules of both aR and bR. But $(a + bf)rR = bfrR \neq 0$ is an essential submodule of bR. Since $bfQ = bfrQ$ there exist $c, d \in R$ with c regular such that $bf = bfr \cdot dc^{-1}$. Thus $(a + bf)c = ac + bfrd$ and so $ac \in (a + bf)R$ and generates an essential submodule of aR. ∎

LEMMA 1.2. *The result of Lemma* 1.1 *holds for semiprime right Goldie rings.*

Proof. With the notation as in Lemma 1.1 it is again sufficient to prove the result for bR uniform. Now $Q = Q_1 \oplus \cdots \oplus Q_n$ with Q_i simple Artinian and by suitable ordering $b \in Q_1$. Let $J = \text{r-ann}(bR)$ and $\bar{R} = R/J$. Since bR is uniform, J is prime (and $\bar{R}$ has full quotient ring isomorphic to Q_1). Thus by Lemma 1.1, there exists $f \in R$ such that $(\bar{a} + \bar{b}\bar{f})\bar{R}$ has the same uniform dimension as $\bar{a}\bar{R} + \bar{b}\bar{R}$. We claim that $a + bf$ solves the lemma. To show this, it is sufficient to show that the uniform dimension of $(a + bf)Q$ equals the uniform dimension of $aQ + bQ$, a fact that is easily checked. ∎

LEMMA 1.3. *Let R be a semiprime right Goldie ring and $R = \sum_1^r a_i R$ with a_1 regular. Then there exist $f_i \in R$ with f_1 regular such that $1 = \sum_1^r a_i f_i$.*

Proof. By Lemma 1.2 there exist $g_i \in R$ such that the element

$$a_2' = a_2 + a_3 g_3 + \cdots + a_r g_r$$

has the same uniform dimension as $\sum_2^r a_i R$. Replace a_2 by a_2' (it is clearly sufficient to prove the lemma in this case). Let $1 = \sum_1^r a_i h_i$ with $h_i \in R$ and choose $k_1, k_2 \in R$ such that $a_1 k_1 = a_2 k_2$ and further that this element generates an essential submodule of $a_2 R$. Now $a_1 h_1 R + a_1 k_1 R$ is an essential right ideal, so by Lemma 1.2 there exists $k \in R$ such that $a_1(h_1 + k_1 k)$ is regular. Thus $h_1 + k_1 k$ is regular and

$$1 = a_1(h_1 + k_1 k) + a_2(h_2 - k_2 k) + a_3 h_3 + \cdots + a_r h_r. \quad ∎$$

2. Stable Range Theorems

In this section several forms of the Stable Range Theorem are proved. The Stable Range Theorem itself is proved for right ideal invariant rings. A right Noetherian ring R is called ***right ideal invariant*** if, given any finitely generated right R-module M and any ideal T, then $\mathrm{Kdim}_R(M \otimes T) \leqslant \mathrm{Kdim}_R M$. It is easily checked that Morita equivalent and homomorphic images of right ideal invariant rings are also right ideal invariant. Discussion of ideal invariance is left until the next section, although both commutative and simple rings are clearly ideal invariant. One consequence of the Stable Range Theorem is that results of [2, Chap. I, 3, Chap. V], concerning the general linear group, hold for ideal invariant rings.

Define an *Asano right order* to be a right Noetherian prime ring for which all the nonzero ideals are progenerators. So in particular a simple right Noetherian ring is right Asano. In the special case of an Asano right order (respectively, a simple right Noetherian ring) the Stable Range Theorem can be strengthened to give the same stability property for the generators of a right ideal (respectively, a torsion module). These two results form the cornerstone of the proofs of Serre's theorem and the Cancellation Theorem in later sections.

All three forms of the Stable Range Theorem follow from the same basic result (Proposition 2.1) and the author apologizes for the resulting untidiness in the statement of that proposition. We start with three definitions about Krull dimension. A module M is said to be *p-critical* for some ordinal $p \geqslant 0$, if $\mathrm{Kdim}\, M = p$ and for any proper factor M' of M, then $\mathrm{Kdim}\, M' < p$. A module is called *critical* if it is p-critical for some ordinal $p \geqslant 0$. Second, if a module M has $\mathrm{Kdim}\, M = r$ then the *r-length* of M, written $l_r(M)$, is defined to be the maximum m, such that there exists a chain of submodules

$$M = M_0 \supset M_1 \supset \cdots \supset M_m = 0,$$

with $\mathrm{Kdim}(M_i/M_{i+1}) = r$. Finally, the *critical length* of a Noetherian module M is the least integer m, such that there exists a chain of submodules

$$M = M_0 \supset M_1 \supset \cdots \supset M_m = 0,$$

where M_i/M_{i+1} is cyclic and critical (the existence of such a chain follows from the fact that M has a critical submodule, [5, Theorem 2.1]). Note that both these last two definitions are generalizations of the length of an Artinian module.

Proposition 2.1. *Let R be a ring and M a Noetherian right R-module with* $\mathrm{Kdim}\, M = r < \mathrm{Kdim}\, R$. *Suppose $a, b \in M$ are such that*

$$\mathrm{Kdim}(M/(aR + bR)) < r,$$

then one of the following occurs:

1. *There exists* $f \in R$ *such that* $\mathrm{Kdim}(M/(a + bf)R) < r$.
2. *There exist an ideal* T *in* R *and a right ideal* $L \supset T$ *such that*

$$\mathrm{Kdim}(R/T) = \mathrm{Kdim}(R/L) = r.$$

Further, there exist submodules $M_1 \supset M_2$ *of* M *with* $\mathrm{Kdim}(M/M_1) < r$ *and* $M_1/M_2 \cong R/T \oplus R/L$.

Proof. Without loss of generality, we may assume that $M = aR + bR$. Consider the short exact sequence

$$0 \to bR \to M \to M/bR \to 0. \qquad (*)$$

The proposition is proved by induction on the critical length of bR. Certainly, if bR is zero the result is trivial. So, suppose that bR has critical length $m + 1$ corresponding to the chain

$$bR = C_0 \supset C_1 \supset \cdots \supset C_{m+1} = 0,$$

and that the proposition holds for any module M' containing elements a' and b' which satisfy the criteria of the proposition and such that the critical length of $b'R \leqslant m$.

Now, $C_m = bgR$ for some $g \in R$ and we have an exact sequence

$$0 \to bR/C_m \to M/C_m \to M/bR \to 0.$$

Assume that $\mathrm{Kdim}(M/C_m) = r$. Then, since the critical length of bR/C_m equals m, the inductive hypothesis applies to the module $M' = M/C_m$ and the elements a and b. If possibility 2 applies to M/C_m, then it also applies to M and we are through. Otherwise, there exists $h \in R$ such that

$$\mathrm{Kdim}(M/(C_m + (a + bh)R)) < r.$$

Thus, by replacing a by $a + bh$ and b by bg, we have reduced the problem to considering the case when bR is critical. We may further assume that bR is r-critical, since otherwise we can obtain possibility 1 by taking $f = 0$. Consider again the sequence $(*)$. Now M/bR is generated by a. If $aR \cap bR$ is nonzero then by the criticality of bR,

$$\mathrm{Kdim}\, M/aR = \mathrm{Kdim}(bR/(aR \cap bR)) < r,$$

and again we have obtained possibility 1 by taking $f = 0$.

Thus, it remains to consider the case when $(*)$ splits. So,

$$M \cong aR \oplus bR \cong R/K \oplus R/L$$

for some right ideals K and L, which are nonzero since Kdim $M <$ Kdim R. Let $T =$ r-ann(R/L) and consider the two possibilities, $T \supseteq K$ and $T \not\supseteq K$. If $T \supseteq K$, then $R/T \oplus R/L \simeq M/M_2$ for some submodule M_2 of M. This is just possibility 2 of the proposition.

If $T \not\supseteq K$, then there exists $f \in R$ such that $bfR \simeq R/L'$ with $L' \not\supseteq K$. Let σ be the homomorphism from $R/K \cap L'$ into $R/K \oplus R/L'$, given by $\sigma(\bar{r}) = (\bar{r}, \bar{r})$ and identify Im(σ) with its image in M. Since bR is critical, so is R/L' and thus

$$\mathrm{Kdim}(R/(K + L') \oplus R/(K + L')) < r.$$

But Im(σ) contains $(L' + K)/K \oplus (L' + K)/L'$ and therefore

$$\mathrm{Kdim}((bfR + aR)/\mathrm{Im}(\sigma)) < r.$$

Thus Kdim$(M/\mathrm{Im}(\sigma)) < r$. Finally, Im$(\sigma)$ is just the submodule of M generated by $a + bf$. Thus, this element gives possibility 1 of the proposition. ▌

The various forms of the Stable Range Theorem are now easy to prove. In each case, it is shown that possibility 2 of Proposition 2.1 cannot occur and then 1 is used to give the result.

THEOREM 2.2 (Stable Range Theorem for Modules). *Let R be a right Noetherian simple ring with* Kdim $R \geqslant n$. *Let M be a finitely generated right R-module with* Kdim $M = n - 1$ *and suppose that $M = \sum_1^{m+1} a_i R$ with $m \geqslant n$. Then there exist $f_i \in R$ such that*

$$M = \sum_1^m (a_i + a_{m+1} f_i)R.$$

Proof. It is clearly sufficient to find $f_{ij} \in R$ such that $M = \sum_1^m a_i' R$, with

$$a_i' = a_i + a_{i+1} f_{ii+1} + \cdots + a_{m+1} f_{im+1} .$$

The a_i' are chosen by induction. Suppose we have found $a_1', \ldots, a_r'$ (possibly $r = 0$) such that if $M_r = \sum_1^r a_i' R$, then Kdim$(M/M_r) \leqslant n - r$. (Note that with this construction of the a_i', M is generated by $a_1', \ldots, a_n'$. Thus for $n + 1 \leqslant i \leqslant m$ take $a_i' = a_i$). Since R is simple, possibility 2 of Proposition 2.1 cannot occur. Thus, by a second induction we can choose elements $b_i \in M$ (where $b_{m+1} = a_{m+1}$ and $b_i = a_i + b_{i+1} g_i$ with $g_i \in R$ and $i \geqslant r + 1$) such that

$$\mathrm{Kdim}(M/(M_r + a_{r+1}R + \cdots + a_{i-1}R + b_i R)) < n - r.$$

Now take $a'_{r+1} = b_{r+1}$. ▌

Remark. In the above proof the simplicity of R is only used to eliminate possibility 2 of Proposition 2.1. Thus, for a more arbitrary ring R the above proof still holds, provided that we can still eliminate this possibility.

LEMMA 2.3. *Let R be an Asano right order, let T be an ideal of R, and let I be a right ideal of R, with* Kdim $R/T = r$. *Then*

$$r\text{-}length(I/IT) \leqslant r\text{-}length(R/T).$$

Proof. Since T is invertible it has the Artin–Rees property; that is, if J is a right ideal of R, there exists $n < \infty$ such that $JT \supseteq J \cap T^n$ (see, for example, [6, Lemma 2.1]). Now consider the diagram

$$\begin{tikzcd}[column sep=small]
 & R \arrow[d, dash] & \\
 & I+T \arrow[dl, dash] \arrow[dr, dash] & \\
I \arrow[dr, dash] & & T \arrow[dl, dash] \\
 & I \cap T \arrow[d, dash] & \\
 & IT \arrow[d, dash] & \\
 & I \cap T^n &
\end{tikzcd}$$

Certainly $\operatorname{Kdim}(I/I \cap T^n) \leqslant r$ so $l_r(I/I \cap T^n) < \infty$. But

$$\begin{aligned} l_r(IT/(I \cap T^n)) &= l_r(IT/(IT \cap T^n)) = l_r((IT + T^n)/T^n) \\ &= l_r((I + T^{n-1})/T^{n-1}), \end{aligned}$$

and $I/I \cap T^n \cong (I + T^n)/T^n$. Thus from the short exact sequence (with the obvious maps),

$$0 \to (I \cap T^{n-1} + T^n)/T^n \to (I + T^n)/T^n \to (I + T^{n-1})/T^{n-1} \to 0$$

we have

$$l_r(I/IT) = l_r((I \cap T^{n-1} + T^n)/T^n) \leqslant l_r(T^{n-1}/T^n) = l_r(R/T). \quad \blacksquare$$

THEOREM 2.4 (Stable Range Theorem for Ideals). *Let R be an Asano right order with* Kdim $R = n$ *and suppose K is a right ideal of R with $K = \sum_1^{m+2} a_i R$ and $m \geqslant n$. Then, there exist $f_i \in R$ such that*

$$K = \sum_1^{m+1} (a_i + a_{m+2} f_i) R.$$

Proof. By Lemma 1.1 choose $a_1' = a_1 + a_2 f_{12} + \cdots + a_{m+2} f_{1m+2}$ such that $\operatorname{Kdim}(K/a_1'R) < n$. The proof of Theorem 2.2 and the remark thereafter complete the proof provided that possibility 2 of Proposition 2.1 can be eliminated.

So, we have an ideal T and a right ideal $L \supset T$ of R, with Kdim $R/T =$ Kdim $R/L = r$ (say) and right ideals $J \subset I \subset K$ such that $I/J \cong R/T \oplus R/L$. In particular $IT \subseteq J$. But Kdim $R/L = r$, so, by Lemma 2.3,

$$l_r(R/T) < l_r(I/J) \leqslant l_r(I/IT) \leqslant l_r(R/T)$$

which gives the required contradiction. ▮

Note that Theorem 2.4 gives, in particular, a bound on the number of generators of a right ideal in an Asano right order. This therefore gives another proof of [16, Theorem 5.4].

THEOREM 2.5 (Stable Range Theorem). *Let R be a right Noetherian right ideal invariant ring with* Kdim $R = n$ *and suppose that* $R = \sum_1^{m+2} a_i R$ *with* $m \geqslant n$. *Then there exist* $f_i \in R$ *such that* $R = \sum_1^{m+1} (a_i + a_{m+2} f_i) R$.

Proof. By Nakayama's lemma, it is sufficient to prove the theorem for semiprime rings. As in the proof of the last theorem, the only nontrivial part is to eliminate possibility 2 of Proposition 2.1.

Keep the notation as in the proof of Theorem 2.4 and notice that we now have the additional information that Kdim$(R/I) < r$. Thus Kdim$(T/IT) =$ Kdim$(R/I \otimes T) < r$ and

$$l_r(R/T) < l_r(I/IT) = l_r(R/IT) = l_r(R/T) + l_r(T/IT) = l_r(R/T).$$

This gives the required contradiction. ▮

We now give a second version of the Stable Range Theorem for Ideals which applies to certain right ideals in overrings of simple rings. This result is a generalization of [18, Proposition 1.7].

THEOREM 2.6. *Let R be a simple right Noetherian ring with* Kdim $R = n$ *and let S be a ring generated by R and elements that centralize R (such that S has the same identity as R). Let K be a right ideal of S such that*

$$K = rS + \sum_1^{m+1} a_i S$$

with $m \geqslant n$ *and r a regular element of R. Then there exist* $f_i \in R$ *such that* $K = rS + \sum_1^m (a_i + a_{m+1} f_i) S$.

Proof. The proof of [18, Proposition 1.7] gives the result, provided that Theorem 2.2 is used to pick the generators of K/rS. ▮

Finally, note that the Stable Range Theorem can be generalized for ideal invariant rings to give the following result, which is just a weaker form of the Stable Range Theorem for Ideals.

COROLLARY 2.7. *Let R be a right Noetherian right ideal invariant ring with* $\mathrm{Kdim}\, R = n$ *and K a right ideal of R with* $\mathrm{Kdim}\, R/K = r$. *Then, there exist $a_1, \ldots, a_{n-r}$ in K such that, if $J = \sum a_i R$, then* $\mathrm{Kdim}\, K/J \leqslant r$. ▌

3. IDEAL INVARIANT RINGS

In this section examples of ideal invariant rings are given. The author knows of no Noetherian ring that is not ideal invariant, nor any general class of right ideal invariant rings that encompasses the examples given below. To check if a ring R is right ideal invariant it suffices to show that, for each cyclic right module M and nonzero ideal T, we have $\mathrm{Kdim}(M \otimes T) \leqslant \mathrm{Kdim}\, M$. If $M \simeq R/I$, then $M \otimes T \simeq T/IT$, so it is sufficient to show that $\mathrm{Kdim}(T/IT) \leqslant \mathrm{Kdim}(R/I)$. This is what is proved in the following results.

PROPOSITION 3.1. *Fully bounded Noetherian rings are ideal invariant.*

Proof. Let I and T be as above. Then I contains an ideal L with $\mathrm{Kdim}\, R/L = \mathrm{Kdim}\, R/I$ [9, Lemma 2.1]. Thus $\mathrm{Kdim}_R(T/IT) \leqslant \text{r-}\mathrm{Kdim}_R(T/LT) = \text{l-}\mathrm{Kdim}_{R/L}(T/LT)$ (by [9, Lemma 2.2]) $\leqslant \text{l-}\mathrm{Kdim}_{R/L}(R/L) = \text{r-}\mathrm{Kdim}_R(R/I)$. ▌

PROPOSITION 3.2. *Semiprime hereditary right Noetherian rings are right ideal invariant. Thus the Stable Range Theorem holds for hereditary right Noetherian rings.*

Proof. The last statement follows from the first by Nakayama's lemma. By [11, Theorem 4.3], a semiprime hereditary right Noetherian ring is a direct sum of prime hereditary rings. But, for a prime hereditary ring R,

$$\mathrm{Kdim}\, R/T \leqslant \mathrm{Kdim}\, R - 1 \leqslant 0.$$

So $\mathrm{Kdim}\, T/IT \leqslant \max(\mathrm{Kdim}\, R/I, \mathrm{Kdim}\, R/T) \leqslant \mathrm{Kdim}\, R/I$. ▌

A ring is said to have *centralizing sets of generators for ideals* if, given any ideal T, there is a set of generators $t_1, \ldots, t_r$ of T such that $t_1 \in C(R)$, the center of R and $t_i + (t_1, \ldots, t_{i-1}) \in C(R/(t_1, \ldots, t_{i-1}))$, for $i = 2, \ldots, r$.

PROPOSITION 3.3. *Let R be a right Noetherian ring with centralizing sets of generators for ideals. Then R is right ideal invariant.*

Proof. Let T be an ideal and let I be a right ideal of R and take $t_1, \ldots, t_r$ to be a centralizing set of generators of T. We prove, by induction on i, that $\mathrm{Kdim}(T_i/IT_i) \leqslant \mathrm{Kdim}(R/I)$, where $T_i = t_1 R + \cdots + t_i R$. Suppose that this is true for T_{i-1} (with $T_0 = 0$). Then the homomorphism from R/I onto

$T_i/(IT_i + T_{i-1})$, given by $1 \mapsto t_i$, is well defined by the centralizing property of t_i mod T_{i-1}. Thus

$$\operatorname{Kdim}(T_i/(IT_i + T_{i-1})) \leqslant \operatorname{Kdim}(R/I).$$

Further, $M = (IT_i + T_{i-1})/IT_i$ is a homomorphic image of T_{i-1}/IT_{i-1} and, so, Kdim $M \leqslant$ Kdim R/I by the inductive hypothesis. Thus $\operatorname{Kdim}(T_i/IT_i) \leqslant$ Kdim R/I. ▮

PROPOSITION 3.4. *If a right Noetherian ring R has centralizing sets of generators for ideals then the same is true for a polynomial extension $R[x]$ of R.*

Proof. Let T be an ideal of $R[x]$ and suppose that T_1 is the biggest ideal contained in T with a centralizing set of generators. Let $f \in T \backslash T_1$ be of minimal degree, say t. Let K be the set of all elements of T of degree t and L the set of leading coefficients of elements of K. Then L is an ideal of R and so has a centralizing set of generators, say $c_1, \ldots, c_n$, which are the leading coefficients of the elements $f_1, \ldots, f_n$ in K. To prove that $T_1 = T$ it is sufficient to show that $f_i \in T$ for $1 \leqslant i \leqslant n$. So, suppose $f_1, \ldots, f_r \in T_1$ and $f_{r+1} \notin T$ (possibly $r = 0$). The maximality of T_1 ensures that $f_{r+1} \notin C(R[x]/T_1)$. So, there exists $g \in R[x]$ with $[f_{r+1}, g] \notin T_1$, where $[a, b] = ab - ba$, and without loss of generality g is a monomial, say ax^m, with $a \in R$. Thus $[f_{r+1}, g] = x^m[f_{r+1}, a]$, so $[f_{r+1}, a] \notin T_1$. But $[c_{r+1}, a] = \sum_1^r c_i k_i$ with $k_i \in R$ and so $[f_{r+1}, a] - \sum_1^r f_i k_i$ has degree $<t$ and thus belongs to T_1. Thus $[f_{r+1}, a] \in T_1$, a contradiction. ▮

4. SIMPLE RINGS

This section proves Serre's theorem and the Cancellation Theorem for simple left Noetherian rings. The first shows that a "big" module over a simple ring has a free direct summand. The second, essentially, gives the uniqueness of the complementary direct summand. First we need to define what is meant by "big." A nonzero element in a right R module is called *torsion* if it is annihilated by some regular element in R. A module M is called *torsion* if every element of M is torsion, and is called *torsion-free* if no nonzero element is torsion. Let $R^{(r)}$ be the free right R-module consisting of the direct sum of r copies of R. The "bigness" of a torsion-free right R-module M is given by its *rank*, rk(M), which, if it exists, is defined to be the least integer r such that M can be embedded in $R^{(r)}$. If R is a semiprime Noetherian ring with full quotient ring Q this is equivalent to $MQ \simeq Q^{(r-1)} \oplus I$ where I is a nonzero right ideal of Q, with possibly $I = Q$ (see, for example, [11, Proposition 1.5]). Note that in this case M has to be finitely generated.

The reason for this definition of rank is because, in proving Serre's theorem and the Cancellation Theorem for right modules over a simple ring R, the only finiteness condition required is that R should be left Noetherian. However, if R is both left and right Noetherian, then Serre's theorem can be proved for arbitrary, finitely generated R-modules (Proposition 4.4). In this case, the criterion of bigness is given by the number of elements needed to generate the module. Both Serre's theorem and the Cancellation Theorem follow easily from Theorem 4.1, the statement of which can be seen to be similar to that of [2, Theorem 9.1]. However, the proof bears no relation to that of Bass.

Notation. Throughout this section we fix the following notation.

R is a simple left Noetherian ring with l-Kdim $R = n$.

Q is the full quotient ring of R.

S is a left Noetherian ring generated by R and elements that centralize R (such that S has the same identity as R).

Let T be any ring. Then elements of $T^{(n)}$ are written in vector notation and ϵ_i denotes the element $(0,..., 0, 1, 0,..., 0)$ with the nonzero term in the ith copy of T. Now the endomorphisms of $T^{(n)}$ can be identified with $n \times n$ matrices over T. In particular ϵ_{ij} denotes the matrix (and corresponding endomorphism) with a 1 in the (i, j)th coordinate and zeros elsewhere.

If α is an element of the right T-module M, write

$$O_M(\alpha) = \{f(\alpha) : f \in \operatorname{Hom}_T(M, T)\}.$$

The suffix is dropped whenever no ambiguity can arise. If $O(\alpha) = T$ then α is called ***unimodular***. This is equivalent to α generating a free direct summand of M. In particular if $M = T^{(n)}$ and $\alpha = (a_1 ,..., a_n)$ then $O(\alpha) = \sum_1^n Ta_i$.

THEOREM 4.1. a) *Let r be an integer with $r \geq n + 2$ and R_1 some simple ring with $R \subseteq R_1 \subseteq S$. Let M be a torsion-free right S-module which is embedded in $S^{(r)}$ in such a way that the subset $M \cap R_1^{(r)}$ has rank r, considered as an R_1-module in the natural way. Let $\alpha = (a_1 ,..., a_r) \in M$ with a_1 regular in R. Then for each $t \in S$ there exists a homomorphism θ from S to M such that*

$$O(\alpha + \theta(t)) = O(\alpha) + St.$$

(b) *If further $\alpha \in M \cap R^{(r)}$, $t \in R$ and $R_1 = R$ then the condition that a_1 be regular can be dropped.*

Remark. It is sufficient to prove that there exists θ such that

$$O(\alpha + \theta(t)) \supseteq St.$$

For suppose this is so and that $\rho(\alpha) \in O(\alpha)$. Then $\rho(\alpha + \theta(t)) = \rho(\alpha) + \rho\theta(t) = \rho(\alpha) + ft$ for some $f \in S$. But $ft \in O(\alpha + \theta(t))$ and, thus, so does $\rho(\alpha)$.

Proof of Theorem 4.1 (a) The method of the proof is to build up the homomorphism θ by a Noetherian induction. Throughout the proof, much use is made of the fact that, since $M \subseteq S^{(r)}$, we have $O(\alpha) \supseteq \sum_1^r Sa_i$.

If $\sum_1^r Sa_i \supseteq St$ then we are through by the above remark. So suppose $\sum_1^r Sa_i \not\supseteq St$. Apply Proposition 2.6 to the left ideal $\sum_1^r Sa_i$ to obtain $f_i \in R$ such that $\sum_1^r Sa_i = \sum_1^{r-1} Sa_i'$ where $a_1' = a_1$ and $a_i' = a_i + f_i a_r$ for $2 \leqslant i \leqslant r-1$. Let σ be the automorphism of $S^{(r)}$ that sends α to the element $(a_1', \ldots, a_{r-1}', a_r)$. More precisely, σ is the automorphism $1_r + \sum_{i=2}^{r-1} \epsilon_{ir} f_i$. It is readily checked that $\sigma(M)$ satisfies the hypotheses given on M in the statement of the theorem. (In particular $\sigma(M) \cap R_1^{(r)}$ still has rank r because the $f_i \in R$ and so σ is also an automorphism on $R_1^{(r)}$.)

Since $\sigma(M) \cap R_1^{(r)}$ has rank r there exists $\epsilon_r f \in \sigma(M)$ with $f \in R_1$. As R_1 is simple, $R_1 f R_1 = R_1$. But $\sum_1^{r-1} Sa_i' = \sum_1^r Sa_i \not\supseteq St$ so there exists $g \in R_1$ such that $\sum_1^{r-1} Sa_i' \not\supseteq Sfgt$. In particular $\sum_1^{r-1} Sa_i' \not\ni fgt$. Let ϕ be the homomorphism from S to $\sigma(M)$ defined by $1 \mapsto \epsilon_r fg$ and write $a_r' = a_r + fgt$. Then

$$\sigma(\alpha) + \phi(t) = \alpha' = (a_1', \ldots, a_r').$$

Since $Sa_r \subseteq \sum_1^{r-1} Sa_i'$ we have $\sum_1^r Sa_i' = \sum_1^{r-1} Sa_i' + Sfgt$. Thus

$$\sum_1^r Sa_i' \supsetneqq \sum_1^{r-1} Sa_i' = \sum_1^r Sa_i .$$

Now if $\sum_1^r Sa_i' \not\supseteq St$ we can repeat the above process on the element α'. By a Noetherian induction we thus obtain an automorphism π of $S^{(r)}$ of the form $1_r + \sum_{i=2}^{r-1} \epsilon_{ir} f_i$ (with $f_i \in R$) and a homomorphism ψ from S to $\pi(M)$ such that if $\pi(\alpha) + \psi(t) = (b_1, \ldots, b_r)$ then $\sum_1^r Sb_i \supseteq St$. Thus if $\theta = \pi^{-1}\psi$ we have

$$O(\alpha + \theta(t)) = O(\pi(\alpha) + \psi(t)) \supseteq \sum_1^r Sb_i \supseteq St$$

as required.

(b) This can be proved by using Theorem 2.4 in place of Proposition 2.6 in the above proof. It is easily checked that at each stage in the proof one can ensure that relevant elements remain inside $R^{(r)}$. ∎

For future reference we need to pinpoint the mechanism in the above proof.

COROLLARY 4.2. *Let R, S, M be as in Theorem* 4.1 (*with $R_1 = R$*). *Then there is an automorphism π of $S^{(r)}$ of the form $1_r + \sum_1^{r-1} \epsilon_{ir} f_i$ with $f_i \in R$ such that $\pi(M)$ satisfies the same hypotheses as M and further $\pi(M) \cap R^{(r)}$ contains an element $\mu = (m_1, \ldots, m_r)$ with m_1 regular in R and $\sum_1^{r-1} Rm_i = R$.*

Proof. This is just what is proved by Theorem 4.1(b) with α arbitrary in

$M \cap R^{(r)}$ and $t = 1$. Further m_1 can be taken to be regular by using Lemma 1.1 to define a second automorphism of $S^{(r)}$. ▮

THEOREM 4.3 (Serre's Theorem). *Let M be a torsion-free right R-module such that $rk(M) = r \geqslant n + 2$. Then $M \simeq M' \oplus R$, for some submodule M'.*

Proof. By the definition of rank, M can be embedded in $R^{(r)}$. Now apply Theorem 4.1(b) with $S = R$, $t = 1$, and α arbitrary. This gives an element $\beta = \alpha + \theta(t) \in M$ with $O(\beta) = R$. Thus β generates a free direct summand. ▮

PROPOSITION 4.4. *Let R be Noetherian with* r-Kdim R *and* l-Kdim $R \leqslant n$. *Let M be a finitely generated right R-module that cannot be generated by less than $2n + 2$ elements. Then $M \simeq M' \oplus R$, for some submodule M'.*

Proof. Let $t(M)$ be the maximal torsion submodule of M. Then $\overline{M} = M/t(M)$ is torsion-free and so can be embedded in $R^{(r)}$ with $r = rk(\overline{M})$. Pick r elements $\bar{\alpha}_1, \ldots, \bar{\alpha}_r$ in $\overline{M}$ such that $\overline{M}/\sum_1^r \bar{\alpha}_i R$ is torsion. Let α_i be the inverse image of $\bar{\alpha}_i$ in M. Then $M/\sum_1^r \alpha_i R$ is torsion. Hence $\text{Kdim}(M/\sum_1^r \alpha_i R) < n$ and so can be generated by n elements (Theorem 2.2). Thus M can be generated by $n + r$ elements and so $r \geqslant n + 2$. By Theorem 4.3 $\overline{M}$ has a free direct summand which pulls back to a free direct summand of M. ▮

THEOREM 4.5 (Cancellation Theorem). *Suppose N is a right R-module which has a torsion-free direct summand of rank $r \geqslant n + 2$. Let P be a finitely generated projective R-module and N' any R-module such that $N \oplus P \simeq N' \oplus P$. Then $N \simeq N'$.*

Proof. Use Theorem 4.1(b) with $S = R$ in the proofs of [2, Theorems 9.2 and 9.3]. ▮

If R is a ring for which Theorem 4.1 holds (with $S = R$) then the Stable Range Theorem for Ideals also holds for left ideals of R. To show this suppose $r \geqslant n + 2$ and let $I = \sum_1^{r+1} Ra_i$ be a left ideal of R. Let M be the right R-module $R^{(r)}$ with $\alpha = (a_1, \ldots, a_r) \in M$ and $t = a_{r+1} \in R$. Then Theorem 4.1(b) gives a homomorphism θ such that

$$I = O(\alpha) + Rt = O(\alpha + \theta(t)) = \sum_1^r R(a_i + f_i a_{r+1})$$

with $f_i \in R$. The converse is not true and an example is given in Section 7 of a ring for which the Stable Range Theorem for Ideals holds, but not Serre's theorem.

4.6. *Morita Equivalence.* For the remainder of this section assume that R is both left and right Noetherian. Serre's theorem and the Cancellation Theorem are Morita invariant in a manner that enables one to obtain, in certain circumstances, a better bound for these theorems. Let I be a projective right ideal of R

and M a torsion-free finitely generated right R-module. If I^* is the dual of I then $\mathrm{End}_R(I) \simeq I \otimes I^*$ and $M \otimes I^*$ is a torsion-free right $\mathrm{End}(I)$-module. Thus if $M \otimes I^*$ has a free direct summand, $M \simeq M \otimes I^* \otimes I$ has a direct summand isomorphic to I.

One can specify further when this occurs. Define the uniform dimension of M, written $u(M)$, in the obvious way; that is, it is the maximum m such that M contains the direct sum of m nonzero submodules. Let $u(R) = y$ and $u(M) = z$, so $\mathrm{rk}(M)$ is the smallest integer greater than or equal to z/y, and suppose that I is a uniform projective right ideal. Now $\mathrm{Kdim}(\mathrm{End}(I)) = n$ and $\mathrm{End}(I)$ is a domain by [15, Theorem 4.5], so $\mathrm{rk}(M \otimes I^*) = z$. Thus M has a direct summand isomorphic to I if $z \geqslant n + 2$, whereas Theorem 4.3 directly gives that M has a free direct summand if $z/y \geqslant n + 2$. Possibly more significantly, if $M \oplus P \simeq N \oplus P$ with P finitely generated and projective, then $M \simeq N$ provided that $z \geqslant n + 2$ (since one can cancel $P \otimes I^*$ from the $\mathrm{End}(I)$-module $(M \oplus P) \otimes I^*$). Thus we have replaced rank by uniform dimension as the criterion of "bigness." Of course if I is not uniform, the above process can still be repeated and it is left to the reader to fill in the appropriate numbers.

5. Polynomial Extensions of Simple Noetherian Rings

In this section Serre's theorem and the Cancellation Theorem are proved for polynomial extensions of simple Noetherian rings. Throughout the section the notation $R[x_1, \ldots, x_m]$ is understood to mean the polynomial extension of R in m commuting indeterminates. The proofs of this section use Theorem 4.1 and thus show that the result is not a trivial extension of the case $S = R$. Further, the results give a better bound than might be expected from the Krull dimension of the ring. So, for example, if R is a simple Noetherian ring with $\mathrm{Kdim}\, R = n$ then Serre's theorem holds for a projective, finitely generated module P over the ring $R[x_1, \ldots, x_m]$, provided that $\mathrm{rk}(P) \geqslant \max(m + 1, n + 2)$.

We start with some general results for which the following definition is required. A ring S is said to be *stably free* if, given any finitely generated projective S-module P, then there exists an integer s such that $P \oplus S^{(s)}$ is free. In K-theoretic terms this says that $K_0(S) = 0$.

Proposition 5.1. (a) *Let S be a left Noetherian left ideal invariant ring with* $\mathrm{Kdim}\, S = m$ *and let $s \geqslant m + 2$ be an integer. Then the automorphisms of the right S-module $S^{(s)}$ are transitive on the unimodular elements of $S^{(s)}$.*

(b) *Suppose further that S is stably free and P is a finitely generated projective right S-module with* $\mathrm{rk}(P) = r \geqslant m + 1$. *Then P is free.*

Proof. (a) By Theorem 2.5, S satisfies the Stable Range Theorem (on the left). Thus this result is just [2, Theorem 4.1a].

(b) There exists an integer s such that $P \oplus S^{(s)} \simeq S^{(r+s)}$. Now part (a) allows one to cancel the extra copies of S. ∎

PROPOSITION 5.2. *Let D be a division ring, $Q = M_s(D)$ a simple Artinian ring, and $S = Q[x_1, \ldots, x_m]$. Define K to be the right ideal of Q with just the first row nonzero and J the right ideal $K[x_1, \ldots, x_m]$ of S. Let P be a finitely generated projective right S-module with uniform dimension $r \geqslant m + 1$. Then $P \simeq J^{(r)}$.*

Proof. If J^* is the dual of J then $\mathrm{End}_s(J) \simeq J \otimes J^* \simeq D[x_1, \ldots, x_m]$. By Proposition 3.4, $D[x_1, \ldots, x_m]$ is ideal invariant and by Grothendieck's theorem, [3, Theorem 3.1, p. 636], $D[x_1, \ldots, x_m]$ is stably free. But $P \otimes J^*$ has rank $r \geqslant m + 1$ over $D[x_1, \ldots, x_m]$ and so is free. Hence $P \simeq P \otimes J^* \otimes J \simeq J^{(r)}$. ∎

LEMMA 5.3. *Let S be a prime Noetherian ring which has the Ore condition with respect to a set $\mathscr{C}$ and let T be the partial quotient ring of S obtained by localization at $\mathscr{C}$. Let M be a finitely generated right S-module with $M \subseteq T^{(r)}$ with r finite. Then there is an automorphism σ of $T^{(r)}$ given by left multiplication by an element of $\mathscr{C}$ such that $\sigma(M) \subseteq S^{(r)}$. If $\alpha \in MT$ then there exist $c, d \in \mathscr{C}$ such that $c\alpha d \in \sigma(M)$.*

Proof. If the generators of M are $\alpha_1, \ldots, \alpha_t$ then choose $c \in \mathscr{C}$ such that $c\alpha_i \in S^{(r)}$ for $1 \leqslant i \leqslant t$. Now σ is left multiplication by c. The second part of the lemma follows from the fact that $\alpha = \beta d^{-1}$ for some $\beta \in M$ and $d \in \mathscr{C}$. ∎

We are now in a position to prove Serre's theorem for polynomial extensions of simple rings. Essentially, the proof just uses Proposition 5.2 to reduce the problem to one that is taken care of by Theorem 4.1.

THEOREM 5.4. *Let R be a Noetherian simple ring with uniform dimension y and 1-Kdim $R = n$. Let $S = R[x_1, \ldots, x_m]$ and P be a finitely generated projective right S-module with $\mathrm{rk}(P) = r \geqslant \max(n + 2, m/y + 1)$. Then $P \simeq P' \oplus S$.*

Proof. Let $\mathscr{C}$ be the regular elements of R and localize S at $\mathscr{C}$ to obtain the quotient ring $T = Q[x_1, \ldots, x_m]$, where Q is the full quotient ring of R. Now $u(PT) = u(P) = z \geqslant m + 1$ so by Proposition 5.2, with the notation as in that proposition, $PT \simeq J^{(z)} \simeq T^{(r-1)} \oplus I$, where I is a nonzero right ideal of T made up of the direct sum of a suitable number of copies of J. Take the above isomorphisms to be equalities and apply Lemma 5.3. This gives an automorphism σ of $T^{(r)}$ such that $\sigma(P) \subseteq S^{(r)}$. Since for $1 \leqslant i \leqslant r - 1$, $PT \ni \epsilon_i$, then $\sigma(P) \ni \epsilon_i r_i$ with r_i regular in R. Likewise $PT \ni \epsilon_r s$ with $s \neq 0 \in Q$, so $\sigma(P) \ni \epsilon_r s_1$ with $s_1 \neq 0 \in R$. Thus $\sigma(P) \cap R^{(r)}$ has rank r as an R-module. Now Theorem 4.1(b), with α arbitrary in $\sigma(P) \cap R^{(r)}$ and $t = 1$, gives the result. ∎

The projectivity condition on P in the above theorem can be weakened to the requirement that PT is a projective T-module. In particular this gives the following corollary (which is also a special case of Theorem 7.2).

COROLLARY 5.5. *Let R be a Noetherian simple ring with* 1-Kdim $R = n$ *and* $S = R[x]$. *Let M be a finitely generated torsion-free right S-module with* $\mathrm{rk}(M) \geqslant n+2$. *Then* $M \cong M' \oplus S$. ∎

THEOREM 5.6. *Let R be a Noetherian simple ring of uniform dimension y and* 1-Kdim $R = n$. *Let $S = R[x_1, \ldots, x_m]$ and P be a finitely generated projective right S-module with* $\mathrm{rk}(P) = r \geqslant 1 + \max(n+2, m/y + 1)$. *If P' and P'' are finitely generated projective S-modules such that $P \oplus P'' \cong P' \oplus P''$, then $P \cong P'$.*

Proof. By [2, Theorems 9.2 and 9.3] it is sufficient to prove the following lemma.

LEMMA 5.7. *Let R, S, and P be as in the theorem. Suppose $t \oplus \beta \in S \oplus P$ is unimodular. Then there exists a homomorphism θ from S to P such that $\beta + \theta(t)$ is unimodular in P.*

Proof. As with Theorem 5.4 the method of the proof is to find an embedding of P into $S^{(r)}$ in such a manner that Theorem 4.1 can be applied.

Let $T = Q[x_1, \ldots, x_m]$ where Q is the full quotient ring of R. Then, up to an isomorphism, Proposition 5.2 gives that $PT = T^{(r-1)} \oplus I$ where, in the notation of Proposition 5.2, I is a nonzero right ideal of T made up of the direct sum of copies of J. Let $\beta = (b_1, \ldots, b_r)$. Then $t \oplus \beta$ is unimodular in $T \oplus PT$ and so

$$T = Tt + \sum_{1}^{r-1} Tb_i + O(b_r).$$

Now by the Stable Range Theorem (Theorem 2.5), there exist $f_i \in T$ such that, if $b_i' = b_i + f_i t$ for $1 \leqslant i \leqslant r-1$, then $T = O(b_r) + \sum_1^{r-1} Tb_i'$. This defines a homomorphism ϕ from T to PT such that $\beta + \phi(t)$ is unimodular in PT. Specifically $\phi(1) = (f_1, \ldots, f_{r-1}, 0)$. Now $PT = (\beta + \phi(t))R \oplus P'$ for some submodule P'. Since $u(P') \geqslant m+1$, Proposition 5.2 gives $P' \cong T^{(r-2)} \oplus I$. Equivalently there exists an automorphism ρ of PT such that $\rho(\beta + \phi(t)) = \epsilon_1$. This is only possible if

$$\rho(\beta) = (1 + g_1 t, g_2 t, \ldots, g_r t)$$

with $g_i \in T$. Now apply Lemma 5.3 to obtain a regular element q in R such that, if π is the automorphism of $T^{(r)}$ given by left multiplication by q, then $\pi(P) \subseteq S^{(r)}$ and $\pi(P) \cap R^{(r)}$ has rank r as an R-module. (The last statement follows from the fact $\epsilon_i \in PT$ for $1 \leqslant i \leqslant r-1$ and $\epsilon_r s \in PT$ for some $s \in Q$.) Further one can pick q such that one also has $qg_i \in S$ for $1 \leqslant i \leqslant r$.

So by replacing P by $\pi\rho(P)$ we can assume that $P \subseteq S^{(r)}$ with $P \cap R^{(r)}$ of rank r as an R-module. Also β has the form $(q + h_1 t, h_2 t, \ldots, h_r t)$ with q regular in R and $h_i \in S$. Thus P has the form required by Theorem 4.1(a). It remains to get β in the form required by that theorem.

Apply Corollary 4.2 to P. This gives an automorphism σ of $S^{(r)}$ (defined by $\sigma = 1_r + \sum_1^{r-1} \epsilon_{ir} k_i$ with $k_i \in R$) such that $\sigma(P) \cap R^{(r)}$ contains an element $\mu = (m_1, \ldots, m_r)$ with m_1 regular and $\sum_1^{r-1} Rm_i = R$. Note that $\sigma(P)$ and $\sigma(\beta)$ still have the same form as was given in the last paragraph.

We next show that we can assume that $\sigma(P) \cap R^{(r)}$ contains $\epsilon_r s$ with s regular in R. Briefly, since $r \geqslant n + 3$, by a second automorphism of $S^{(r)}$ (defined by the Stable Range Theorem) we can assume that $\sum_1^{r-2} Rm_i = R$. Certainly $\sigma(P) \cap R^{(r)}$ contains $\epsilon_{r-1} s$ with s regular in R and so a third automorphism of $S^{(r)}$ puts this element into the end position. Notice that neither of these two automorphisms affects the form of $\sigma(P)$ or $\sigma(\beta)$.

So $\epsilon_r s \in \sigma(P) \cap R^{(r)}$ with s regular in R. By Lemma 1.1 there exists $s_1 \in R$ such that $(1 - m_r) + ss_1$ is regular in R. Replace m_r by $m_r - ss_1$. By Lemma 1.3 choose $f_i \in R$, for $1 \leqslant i \leqslant r - 1$, such that f_1 is regular and $\sum_1^{r-1} f_i m_i = 1$. Let τ be the automorphism of S sending m to $m' = (m_1, \ldots, m_{r-1}, 1)$; i.e., τ is the automorphism $1_r + \sum_1^{r-1} \epsilon_{ri}(1 - m_r) f_i$. Now $\tau\sigma(P)$ still satisfies the hypotheses of Theorem 4.1(a) and

$$\tau\sigma(\beta) = (q + h_1 t, h_2 t, \ldots, h_{r-1} t, q' + h_r' t),$$

where $h_r' \in S$ and $q' = (1 - m_r) f_1 q$ which is regular in R by construction. Define ϕ to be the homomorphism from R to $\tau\sigma(P)$ sending 1 to $m' h_r'$. Then let

$$\alpha = \tau\sigma(\beta) + \phi(t) = (k_1, \ldots, k_{r-1}, q')$$

for some $k_i \in S$. Now finally apply Theorem 4.1(a) to the element $\alpha \in \tau\sigma(P)$ and $t \in S$. ∎

For the case of polynomial extensions of simple rings we have been able to obtain a better bound for both Serre's theorem and the Cancellation Theorem than might have been expected from the Krull dimension of the ring. It is worth noting that this also occurs for polynomial extensions of suitable commutative rings, and for the best results in that case the reader is referred to [20]. We can also use Lemma 5.7 to obtain a better bound for the Stable Range Theorem.

COROLLARY 5.8. *Let R be a Noetherian simple ring of uniform dimension y and* 1-Kdim $R = n$. *Let $S = R[x_1, \ldots, x_m]$ and suppose that $S = \sum_1^{r+1} Sa_i$ with $r \geqslant 1 + \max(n + 2, m/y + 1)$. Then there exist $f_i \in S$ such that*

$$S = \sum_1^r S(a_i + f_i a_{r+1}).$$

Proof. Apply Lemma 5.7 to the right module $P = S^{(r)}$ with $\beta = (a_1, \ldots, a_r)$ and $t = a_{r+1}$. ∎

In this section we have had to assume a projectivity condition. This is in general neccessary. For suppose "projective" can be replaced by "torsion-free" in Theorem 5.4. Let I be a left ideal of $S = R[x_1, ..., x_m]$, with $m \geqslant 2$, such that I cannot be generated by as few as $m + 4$ elements. Let $a_1, ..., a_r$ be a minimal generating set for I and take $\alpha = (a_1, ..., a_r) \in S^{(r)}$ (considered as a right S-module). Let N be the torsion submodule of $M = S^{(r)}/\alpha S$. Then M/N is torsion-free of rank $r - 1 \geqslant m + 3$ and so has a free direct summand which pulls back to a free direct summand of $S^{(r)}$. Thus $S^{(r)} = P \oplus \gamma S$ with γ unimodular and $\alpha \in P$. But $P \simeq S^{(r-1)}$ by Theorem 5.6, so let $\beta = (b_1, ..., b_{r-1})$ be the image of α in $S^{(r-1)} \simeq P$. Then

$$I = \sum_1^r Sa_i = O(\alpha) = O(\beta) = \sum_1^{r-1} Sb_i$$

and so can be generated by $r - 1$ elements, a contradiction.

The final result of this section is a second type of Cancellation Theorem. Much of this section has been concerned with the ring $S = R[x_1, ..., x_m]$ with R simple. This next result says that R is uniquely defined (up to isomorphism) by S. In the notation of [1], simple rings are invariant.

Proposition 5.9. *Let R, T be rings such that R is simple and*

$$S = R[x_1, ..., x_m] \simeq T[y_1, ..., y_m].$$

Then $R \simeq T$.

Proof. Assume that $S = R[x_1, ..., x_m] = T[y_1, ..., y_m]$. Then the center of S is $k[x_1, ..., x_m] = h[y_1, ..., y_m]$ where k and h are the centers of R and T, respectively. Thus k is a field and so, being the units of $k[x_1, ..., x_m]$, is contained in h. Thus by [1, 1.1], $k = h$. So suppose $y_i = f_i(x) \in k[x_1, ..., x_m]$. Clearly k and the f_i generate $k[x_1, ..., x_m]$. So let σ be the automorphism of S which sends x_i to $f_i(x) = y_i$ and fixes R. Then

$$R[y_1, ..., y_m] = \sigma(R[x_1, ..., x_m]) = T[y_1, ..., y_m]$$

and so $R \simeq S/(y_1, ..., y_m) \simeq T$. ∎

6. Weyl Algebras

In the last section it was shown that better bounds were obtained, for Serre's theorem and the Cancellation Theorem for a polynomial extension of a simple ring, than might have been expected by the Krull dimension of the ring. In this section the process is repeated for the Weyl algebras. The Weyl algebra $A_n(D)$ is defined to be the associative D-algebra with 1 (where D is always a division ring of characteristic zero) generated by the $2n$ elements $x_1, ..., x_n$, $\theta_1, ..., \theta_n$ subject to the relations $[x_i, \theta_j] = \delta_{ij}$ and all the other commutators being zero. In particular we prove that the bounds obtained for $A_n(D)$ are independent of the Krull dimension of the ring (which, for D a field, is equal to n).

THEOREM 6.1. *Let M be a finitely generated torsion-free $A_n(D)$-module with* $\mathrm{rk}(M) = r \geqslant 4$. *Then $M \simeq M' \oplus A_n$.*

Proof. The result is proved by induction on n. If $n = 1$ then

$$\mathrm{Kdim}(A_1(D)) \leqslant 2$$

by [14] and, since $A_n(D)$ is simple, the result is just Theorem 4.3. So suppose the theorem is true for $A_{n-1}(D')$ for any division ring D' of characteristic zero. Identify A_1 with the subring $D[x_1, \theta_1]$ of A_n and let D' be the full quotient ring of A_1. Then $D' \otimes_D A_n(D) \simeq A_{n-1}(D')$. So MD' is a torsion-free $A_{n-1}(D')$-module of rank $r \geqslant 4$. Thus by induction

$$MD' \simeq A_{n-1}(D') \oplus N' \subseteq A_{n-1}(D')^{(r)},$$

where N' is a torsion-free $A_{n-1}(D')$-module. Take the isomorphism to be an equality and apply Lemma 5.3. With σ defined as in that lemma, $\sigma(M) \subseteq A_n(D)^{(r)}$ and since $MD' \ni \epsilon_1$, then $\sigma(M) \ni \epsilon_1 q$ with q some element of A_1. Now apply Theorem 4.1(a) with $\alpha = \epsilon_1 q$, $R = A_1(D)$, $R_1 = S = A_n(D)$, and $t = 1$. ∎

THEOREM 6.2. *Let M be a finitely generated torsion-free $A_n(D)$-module with* $\mathrm{rk}(M) = r \geqslant 5$. *Let $\alpha \in M$ and $t \in A_n$. Then there exists a homomorphsim θ from A_n to M such that $O(\alpha + \theta(t)) = O(\alpha) + A_n t$.*

Proof. Let N be the torsion submodule of $\bar{M} = M/\alpha A_n$. Then $\bar{M}/N$ is a torsion-free A_n-module of rank $r - 1 \geqslant 4$. Thus it has a free direct summand, which pulls back to a free direct summand of M. Hence, $M = M' \oplus \gamma A_n$ with γ unimodular and $\alpha \in M'$. Let θ be the homomorphism sending 1 to γ. Then

$$O_M(\alpha + \theta(t)) = O_{M'}(\alpha) + A_n t = O_M(\alpha) + A_n t. \quad ∎$$

THEOREM 6.3. *Let N be a right $A_n(D)$-module such that $N = N' \oplus M$ where M is a torsion-free finitely generated right A_n-module with* $\mathrm{rk}(M) \geqslant 5$. *Suppose L and P are A_n-modules with P finitely generated, projective, and $N \oplus P \simeq L \oplus P$. Then $N \simeq L$.*

Proof. Use Theorem 6.2 in [2, Theorems 9.2 and 9.3]. ∎

COROLLARY 6.4. *Let P be a projective finitely generated $A_n(D)$-module with* $\mathrm{rk}(P) = r \geqslant 5$. *Then P is free.*

Proof. By [19, Theorem 2.2] $A_n(D)$ is stably free and so there exists an integer s such that $P \oplus A_n^{(s)} \simeq A_n^{(r+s)}$. Now use Theorem 6.3. ∎

THEOREM 6.5. *Let I be a left ideal of $A_n(D)$ and suppose $I = \sum_1^{r+1} A_n a_i$ with $r \geqslant 5$. Then there exist $f_i \in A_n$ such that $I = \sum_1^r A_n(a_i + f_i a_{r+1})$.*

Proof. Take $\alpha = (a_1, \ldots, a_r)$ in the right module $A_n^{(r)}$ and let $t = a_{r+1}$. Now apply Theorem 6.2 to obtain a homomorphism θ and $f_i \in A_n$ such that

$$I = O(\alpha) + A_n t = O(\alpha + \theta(t)) = \sum_1^r A_n(a_i + f_i a_{r+1}). \quad \blacksquare$$

COROLLARY 6.6 *Any left (or right) ideal of $A_n(D)$ can be generated by just five elements.* $\blacksquare$

If k is a field of characteristic zero then many of the properties of $A_n(k)$ parallel those of $k[x_1, \ldots, x_n]$ (see [12, 14, 17]). Given the validity of Serre's question, which asks whether all projective $k[x_1, \ldots, x_n]$-modules are free, then Corollary 6.4 gives another similarity between the two rings. Presumably the bound of five in the above theorems is not the best possible. Two would seem the most sensible alternative. Certainly the bound cannot be one since there are projective right ideals in A_n which are not free (for example, the right ideal of $A_1(k)$, generated by $x_1{}^2$ and $x_1\theta_1 + 1$, is projective but not free). However the rank 1 case is frequently an exceptional case. For instance there exist projective right ideals that are not free in the ring $D[x_1, x_2]$ where D is any noncommutative division ring [13, Proposition 1].

Note that Proposition 5.1 and [19] prove that, if P is a projective $A_n(k)$-module with $\operatorname{rk}(P) \geqslant n+1$, then P is free. Therefore this generalizes [21, Theorem 3], which proved this for A_1.

7. Asano Orders

In this section we parallel the results about simple rings from Section 4. However, for Asano orders, a somewhat different style of proof is required and no information is obtained about overrings of Asano orders. As in Section 4, the "bigness" of a module is defined in terms of its rank although the comments in 4.6. Morita Equivalence, also apply to this case. Thus in the special case of Dedekind prime rings, where there always exist projective uniform right ideals, one can work entirely with uniform dimension. As always the important result is the following.

THEOREM 7.1. *Let R be an Asano order with* 1-Kdim $R = n$ *and let M be a finitely generated torsion-free right R-module with* $\operatorname{rk}(M) = r \geqslant n+3$. *Then for any $\alpha \in M$ and $t \in R$, there exists a homomorphism θ from R to M such that $O(\alpha + \theta(t)) = O(\alpha) + Rt$.*

Proof. Let $D = \bigcap R_p$ where p runs through all maximal ideals of R. By

[6, Proposition 2.4 and Theorem 3.5], this is well defined and D is a bounded Asano order and so by [10] is a Dedekind prime ring. By [4, Theorem 2.4], up to an isomorphism, $MD = I \oplus D^{(r-1)}$ where I is a nonzero right ideal of D.

Choose a homomorphism ϕ from R to M such that, if $\beta = \alpha + \phi(t)$, then $O(\beta)$ is maximal. If $O(\beta) \supseteq Rt$ we are through by the remark after Theorem 4.1. So suppose that $O(\beta) \not\supseteq Rt$. Now $O(\beta)$ is a left ideal of R and so can be generated by $n+1$ elements, by Theorem 2.4. Thus there exist maps $\psi_1, \ldots, \psi_{n+1}$ from M to R such that $O(\beta)$ is generated by the $\psi_i(\beta)$. Now the ψ_i define unique maps from MD to D and so identify the ψ_i with these maps. Thus for each i,

$$\psi_i \in (MD)^* \cong \operatorname{Hom}(MD, D)$$

which is a torsion-free left D-module of rank r. Let N be the torsion submodule of $L = (MD)^*/(\sum_1^{n+1} D\psi_i)$. Then L/N is torsion-free of rank $\geqslant 2$ (since MD^* has rank $r \geqslant n+3$). Thus it has a free direct summand by [4, Theorem 2.4], which pulls back to a free direct summand of $(MD)^*$. Thus $(MD)^* = N \oplus D\zeta$ for some submodule N and unimodular element ζ in $(MD)^*$. Further $\psi_i \in N$ for $1 \leqslant i \leqslant n+1$. Since MD is projective, $MD \cong MD^{**}$. So there exists $\delta \in MD$ such that $\zeta(\delta) = 1$ and $\psi(\delta) = 0$ for any $\psi \in N$.

$O(\beta) \not\supseteq Rt$ so let T be the largest ideal of R such that $O(\beta) \supseteq Tt$ (possibly $T = 0$). Let p be a maximal ideal of R such that $T \subseteq p$. Then for any $c \in \mathscr{C}(p)$ clearly $O(\beta) \not\supseteq (RcR)t$. Let $m_1, \ldots, m_s$ be the generators of M. Thus by Lemma 5.3, with $\mathscr{C} = \mathscr{C}(p)$, there exists $b \in \mathscr{C}(p)$ such that $b\zeta(m_i) \in R$ for $1 \leqslant i \leqslant s$. Further choose $c \in \mathscr{C}(p)$ such that $\delta c \in M$. Now $bc \in \mathscr{C}(p)$ and so $RbcRt \not\subseteq O(\beta)$. Thus there exists $d \in R$ such that $Rbcdt \not\subseteq O(\beta)$ and in particular $bcdt \notin O(\beta)$.

Now let χ be the homomorphism from R to M defined by $\chi(1) = \delta cd$ and let $\gamma = \beta + \chi(t) = \beta + \delta cdt$. Consider $O(\gamma)$. Since by construction $\psi_i(\delta) = 0$, for $1 \leqslant i \leqslant n+1$, we have $\psi_i(\gamma) = \psi_i(\beta)$. Thus $O(\gamma) \supseteq O(\beta)$ and so in particular $b\zeta(\beta) \in O(\gamma)$. Thus $b\zeta(\delta cdt) = bcdt \in O(\gamma)$. However $bcdt \notin O(\beta)$ and so $O(\gamma) \supsetneqq O(\beta)$ which contradicts the maximality of $O(\beta)$. ∎

THEOREM 7.2. *Let R be an Asano order with* 1-Kdim $R = n$ *and M a finitely generated torsion-free right R-module with* $\operatorname{rk}(M) \geqslant n+3$. *Then $M \cong M' \oplus R$.*

Proof. Let $t = 1$ and α be arbitrary in Theorem 7.1. ∎

THEOREM 7.3. *Let R be an Asano order with* 1-Kdim $R = n$ *and N a right R-module with a torsion-free, finitely generated direct summand of rank* $\geqslant n+3$. *Let L and P be right R-modules with P finitely generated, projective, and $N \oplus P \cong L \oplus P$. Then $N \cong L$.*

Proof. Use Theorem 7.1 in the proofs of [2, Theorems 9.2 and 9.3]. ∎

In the special case of a Dedekind prime ring Theorem 7.3 gives a partial answer to the question in [4, Sect. 2] which asked, in our notation, whether the Cancellation Theorem held for such a ring. However, in this special case we can improve the bound to $n+1$.

THEOREM 7.4. *Let R be a Dedekind prime ring and N a right R-module with a finitely generated torsion-free direct summand of uniform dimension $\geqslant 2$. Let L and P be right R-modules with P finitely generated, torsion-free and $N \oplus P \cong L \oplus P$. Then $N \cong L$.*

Proof. Following 4.6, Morita Equivalence, the problem can be reduced to one about Dedekind domains. Spceifically let I be a uniform right ideal of R. Since R is hereditary, I is projective and $\mathrm{End}_R(I) \cong I \otimes I^*$ is a Dedekind domain by [15, Theorem 4.5]. Thus as $\mathrm{End}_R(I)$-modules,

$$(N \otimes I^*) \oplus (P \otimes I^*) \cong (L \otimes I^*) \oplus (P \otimes I^*).$$

So if the theorem holds for domains, $N \otimes I^* \cong L \otimes I^*$ and hence $N \cong L$.

So by [2] it is sufficient to prove Theorem 7.1 when R is a Dedekind domain and $\mathrm{rk}(M) \geqslant 2$. Keep the notation as in Theorem 7.1.

By the remark after Theorem 4.1, it is sufficient to prove that

$$O(\alpha + \theta(t)) \supseteq Rt.$$

Now by [4, Theorem 2.4], up to an isomorphism, $M = I \oplus R^{(r-1)}$ with I a nonzero right ideal of R. Let $\alpha = (a_1, \ldots, a_r) \in I \oplus R^{(r-1)}$ and consider the left ideal $J = \sum_1^r Ra_i + Rt$. Choose any $x \in I$ such that $a_1' = a_1 + xt \neq 0$. Then certainly a_1' generates an essential submodule of J. Thus by Theorem 2.4 and its proof, J is generated by $a_1', a_2, \ldots, a_{r-1}, a_r'$ where $a_r' = a_r + yt$ for some $y \in R$. Now let θ be the homomorphism from R to M sending 1 to $\epsilon_1 x + \epsilon_r y$. Then $\alpha + \theta(t) = (a_1', a_2, \ldots, a_{r-1}, a_r')$ and so $O(\alpha + \theta(t)) \supseteq J \supseteq Rt$. ∎

Define a ring R to be a PIR if every right and left ideal of R is principal. It is well known that $M_n(R)$ being a PIR does not imply that R is a PIR (for example if $R = A_1$, the first Weyl algebra, R is not a PIR but, for $n > 1$, $M_n(R)$ is by [8, Proposition 2.11, p. 51]). However, the following proposition shows that this situation really only occurs if R is a domain.

PROPOSITION 7.5. *Let R be a prime ring such that $S = M_n(R)$ is a* PIR. *Then R is either a* PIR *or a domain. In either case $M_m(R)$, for $m > 1$, is a* PIR.

Proof. By [8, Proposition 2.11], given any domain D, then $M_n(D)$ is a PIR if an only if, given nonzero right (or left) ideals $I_1, \ldots, I_n$ of D, then $I_1 \oplus \cdots \oplus I_n$ is free. But $M_n(D)$ and hence D are Dedekind prime rings by [15, Theorem 3.3]. So $I_1 \oplus I_2 \oplus D^{(n-2)}$ is free and hence, by Theorem 7.4 so is $I_1 \oplus I_2$. Thus by [8] again, $M_2(D)$, and also $M_m(D)$ for $m > 1$, is a PIR. Thus if R is a domain we are through.

So suppose R is not a domain. Then by [8, p. 45], $S \simeq M_r(D)$ for some integer r and domain D. Let I and J be essential right (or left) ideals of R. Now R is Morita equivalent to D, so let I' and J' be the images of I and J under this equivalence. Then I' and J' have uniform dimensions greater than 1 and so are isomorphic by the above. Thus I and J are isomorphic and so must in particular be isomorphic to R. Thus R is a PIR. Furthermore $M_m(R)$ for $m > 1$ is a PIR by [7, Theorem 40]. ▮

COROLLARY 7.6. *Let R be a ring such that $S = M_n(R)$ is a* PIR. *Then for any $m > 1$, $M_m(R)$ is a* PIR.

Proof. By [8, p. 45], S is the direct sum of (non-Artinian) prime PIR's and Artinian primary PIR's. Thus it is sufficient to prove the theorem for these two separate cases, the first of which is proved by Proposition 7.5. So suppose $S = M_n(R)$ is an Artinian primary PIR. Then R is primary Artinian and so by [7, Theorem 31] is a full matrix ring over a completely primary ring, say D. Then D is a PIR by [7, Theorem 39]. Thus any full matrix ring over R is a full matrix ring over D and so is a PIR by [7, Theorem 40]. ▮

We observed in Section 4 that a ring that satisfies Theorem 4.1 (or 7.1) also satisfies the Stable Range Theorem for Ideals. It is worth noting that the converse is not true. To show this let Z denote the integers and

$$R = \begin{pmatrix} Z & 2Z \\ Z & Z \end{pmatrix}.$$

Then R is a hereditary, Noetherian, fully bounded prime ring. It also satisfies the Stable Range Theorem for Ideals. (For any right ideal I of R is a submodule of $Z^{(4)}$ as a Z-module and hence is free. Thus as a Z-module and hence as an R-module, I satisfies the Stable Range Theorem with bound at most 5.) However R does not satisfy Serre's theorem. For

$$I = \begin{pmatrix} Z & 2Z \\ Z & 2Z \end{pmatrix}$$

is an idempotent ideal of R and so is not a generator. Thus the direct sum of n copies of I, which has rank n, is not a generator and so cannot have a free direct summand.

ACKNOWLEDGMENTS

The results of this paper form part of the author's Ph.D. thesis. I would like to thank J. C. Robson for his considerable help and encouragement, and the Science Research Council for their financial support.

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